

An algebro-geometric construction of lower central series of associative algebras

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Abstract

The lower central series invariants M_k of an associative algebra A are the two-sided ideals generated by k -fold iterated commutators; the M_k provide a filtration of A . We study the relationship between the geometry of $X = \operatorname{Spec} A_{ab}$ and the associated graded components N_k of this filtration. We show that the N_k form coherent sheaves on a certain nilpotent thickening of X , and that Zariski localization on X coincides with noncommutative localization of A . Under certain freeness assumptions on A , we give an alternative construction of N_k purely in terms of the geometry of X (and in particular, independent of A). Applying a construction of Kapranov, we exhibit the N_k as natural vector bundles on the category of smooth schemes.

1 Introduction

In [Kap98], Kapranov proposes the following viewpoint on non-commutative algebraic geometry. Given a non-commutative algebra A with abelianization A_{ab} , the natural surjection $A \twoheadrightarrow A_{ab}$ can be understood as an inclusion $X = \operatorname{Spec}(A_{ab}) \hookrightarrow \operatorname{Spec}(A)$, where $\operatorname{Spec}(A)$ is the (would-be) spectrum of the algebra A . Kapranov then proposes to study so-called NC-complete algebras: the NC-completion of an algebra A is the “formal neighborhood” to the embedding $X \hookrightarrow \operatorname{Spec}(A)$; thus considering NC-complete algebras A amounts to studying formal thickenings of X into a non-commutative scheme.

A basic invariant of a non-commutative algebra A is its lower central series, the descending filtration $L_\bullet = L_\bullet(A)$ by Lie ideals, defined recursively as follows:

$$L_1 := A, \quad L_k := [A, L_{k-1}].$$

In other words, L_k is spanned by iterated commutators, $[a_1, [a_2, [\cdots [a_{k-1}, a_k] \cdots]]$, for $a_i \in A$. The lower central series ideals are $M_k := AL_kA = AL_k$, and the associated graded components are $N_k := M_k/M_{k+1}$. In particular, we have $N_1 = A_{ab} = \mathcal{O}(X)$. In fact, the components N_k depend only on the NC-completion of A , so they are really invariants of Kapranov’s formal neighborhood of X in A , rather than of A itself.

For several years, evidence has been mounting that – despite their origin in non-commutative algebra – the associated graded components N_k , and their Lie analogs $B_k := L_k/L_{k+1}$, should

admit descriptions in terms of the *commutative* algebraic geometry of $X = \text{Spec}(A/M_2)$. The first such evidence was provided by Feigin and Shoikhet [FS07] for the free algebra, A_n , on n generators. They constructed an isomorphism $A_n/M_3 \cong \Omega^{ev}(\mathbb{A}^n)$, identifying the first two components $N_1 \oplus N_2$ with the space of even-degree differential forms on $X = \mathbb{C}^n$, with Fedosov quantized product (see Example 2.10). In [DE08] and [EKM09], it is shown more generally that the components $B_k(A_n)$ and $N_k(A_n)$, respectively, are finite extensions of tensor field modules on \mathbb{A}^n , with respect to a natural action of the Lie algebra W_n of polynomial vector fields. The papers [AJ10], [BJ13a], [BB11], [BEJ⁺12], [BJ13b], [Ker13], all apply geometric techniques echoing [FS07] to compute $B_k(A)$ and $N_k(A)$ in various special cases.

The present paper aims to explain the geometric nature of the lower central series filtration in two essential ways: Firstly, we prove that the components $N_k(A)$ are Zariski local on X , and are in fact coherent sheaves on a certain canonical nilpotent thickening of X . Secondly, under the assumption that A is locally free – in particular, for Kapranov’s NC-manifolds, and for formally smooth (a.k.a quasi-free) algebras of Cuntz-Quillen [CQ95] – we give an alternative construction of the lower central series in terms of the formal geometry of X . This construction implies that $N_k(A)$ is not only local on X , but it is in fact formal: $N_k(A)$ is completely determined by its value in a formal neighborhood of any point $x \in X$, which in turn has a universal answer depending only on the dimension of X .

An outline of the paper is as follows. In Section 2.2, for an arbitrary finitely generated algebra A , we begin by introducing a commutative product on the quotient A/M_3 as follows:

Proposition 2.6. The star-product, $a \star b := \frac{1}{2}(ab + ba)$, is associative and commutative on $A_\star := (A/M_3, \star)$, and makes each $N_k(A)$ into a finitely generated A_\star -module.

Kapranov has shown in [Kap98] that NC-complete algebras A admit flat Ore localizations with respect to multiplicative subsets $S \subset A/M_2$ without zero divisors. Our first main result – the content of Section 3 – is that Ore localization of an algebra A coincides with the ordinary, *commutative* localization on the A_\star -module $N_k(A)$:

Theorem 3.1. Fix a multiplicative subset $\bar{S} \subset (A/M_2)$ without zero divisors, and let $S := \pi^{-1}(\bar{S}) \subset A$. The map,

$$m : A_\star[S^{-1}] \otimes_{A_\star} N_k(A) \xrightarrow{\sim} N_k(A[S^{-1}]),$$

$$a \otimes n \mapsto an,$$

is an isomorphism.

The significance of this theorem is that it allows us to work with $N_k(A)$ as a coherent sheaf $\widetilde{N}_k(A)$ on X_\star , applying tools from commutative algebraic geometry, such as localizations, completions, and Hilbert’s syzygy theorem, to perform computations. Let \widehat{A}_n denote the degree completion of the free algebra A_n . As applications of Theorem 3.1 we have:

Corollary 3.9. The completion $N_k(A)_{(\mathfrak{m}_x)}$ at any smooth point $x \in X^{sm}$ is a quotient of $N_k(\widehat{A}_n)$, where $n = \dim X$.

Corollary 3.10. Suppose that $X = \text{Spec}(A/M_2)$ is a zero or one dimensional scheme with finitely many non-reduced points. Then each $N_k(A)$ is finite dimensional, for $k \geq 2$.

Corollary 3.12. Suppose A is the quotient of a free algebra on generators x_1, \dots, x_n in degrees d_1, \dots, d_n , by a collection of homogeneous non-commutative polynomials, so that A is graded by degree. Then the Hilbert series for each $N_k(A)$ is a rational function, with poles at d_i th roots of unity.

In Section 4, we turn our attention to locally free algebras.

Definition 1.1. An algebra A is called *locally free of rank n* if, for every $x \in X$, there exists an isomorphism $A_{(\mathfrak{m}_x + M_2)} \cong \hat{A}_n$.

Example 1.2. There are many examples of locally free algebras:

- (i) The free algebra A_n is locally free of rank n .
- (ii) The coordinate algebra $\mathcal{O}(X)$ of any connected smooth affine curve X is locally free of rank one.
- (iii) The free product of locally free algebras of ranks m and n is locally free of rank $m + n$.
- (iv) For any smooth complete intersection $X = \mathbb{C}[x_1, \dots, x_{n+m}]/\langle f_1, \dots, f_m \rangle$, the algebra $A = A_{n+m}/\langle \tilde{f}_1, \dots, \tilde{f}_m \rangle$, where each $\tilde{f}_i \in f_i + M_2$, is locally free of rank $\dim X = n$.
- (v) If A is locally free, and if \bar{S} is any multiplicative subset of A/M_2 without zero divisors, then $A[S^{-1}]$ is locally free, where $S = \pi^{-1}(\bar{S})$.
- (vi) Any NC-smooth algebra (see [Kap98], Proposition 1.5.1), including Kapranov's NC-smooth thickenings of any finitely generated smooth commutative algebra (see [Kap98], Proposition 1.6.1), is locally free.
- (vii) Any formally smooth (a.k.a quasi-free) algebra is locally free, by the formal tubular neighborhood theorem ([CQ95], Section 6, Theorem 2).

Remark 1.3. We have isomorphisms:

$$(A/M_2)_{(\mathfrak{m}_x)} \cong A_{(\mathfrak{m}_x + M_2)}/M_2 \cong \hat{A}_n/M_2 \cong \mathbb{C}[[x_1 \dots x_n]].$$

Thus if A is locally free of rank n , then X is smooth of dimension n .

Under these assumptions, we give a completely new construction of $N_k(A)$ in the language of formal geometry on X . Given a module M over the Lie algebra W_n , which satisfies certain finiteness conditions, we have the globalization $\mathcal{GL}_X(M)$, a sheaf on X which “spreads” M around X , using the W_n -action to produce transition functions (see Section 4.3 for a review of formal geometry). Our second main result is:

Theorem 4.26. We have $\widetilde{N}_k(A) \cong \mathcal{GL}_X(N_k(\hat{A}_n))$ as vector bundles on X . In particular, we have $N_k(A) \cong \Gamma(X, \mathcal{GL}_X(N_k(\hat{A}_n)))$, as $U(\text{Vect}(X)) \rtimes O(X)$ -modules.

Combining Theorems 3.1 and 4.26, we exhibit N_k as a *natural vector bundle* - a functorial assignment to any smooth scheme X of a vector bundle over X ; remarkably this characterization is completely independent of the choice of locally free lift A . It is well known that the category of natural vector bundles is equivalent to the category of finite dimensional representations of vector fields on a formal disc vanishing at the origin; this allows us to reduce the problem of describing the components N_k for all locally free algebras of a given rank to a single, finite dimensional problem in representation theory; some example computations are given at the end of Section 4.3.

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2 Preliminaries

For the remainder of the paper, A denotes a finitely generated associative algebra, and $X := \text{Spec}(A/M_2(A))$.

2.1 NC-nilpotent algebras and localizations

In this subsection, we compare the filtration of A by lower central series ideals to Kapranov's NC-filtration [Kap98].

Definition 2.1. We consider the following filtrations on A :

1. The lower central series filtration L_\bullet is defined recursively by $L_1 = A, L_k := [A, L_{k-1}]$.
2. The associative lower central series filtration M_\bullet is defined by $M_k := AL_kA = AL_k$.
3. The NC-filtration F_\bullet is defined by $F_k = \sum_{i_1+\dots+i_m=k-m} M_{i_1} \cdots M_{i_m}$.

We denote by $B_k := L_k/L_{k+1}, N_k := M_k/M_{k+1}$ the associated graded components of each filtration.

Remark 2.2. In Sections 3 and 4 we apply geometric techniques to the graded components of the filtration M_k . The same results in fact hold for the associated graded components of F_k , with easier proofs (compare, for example, Theorem 3.1 with Theorem 2.1.6 of [Kap98]).

Definition 2.3. An algebra A is *Lie complete* (resp, *NC-complete*) if it is complete in the topology induced by the filtration M_\bullet (resp, F_\bullet). Similarly, A is *Lie nilpotent* (resp, *NC-nilpotent*) if $M_N(A) = 0$ (resp, $F_N(A) = 0$) for some N .

Clearly, we have that $M_k \subset F_{k-1}$. Conversely, ([Jen47], Theorem 2) proves that for any finitely generated algebra, and any k , there exists N such that $M_2^N \subset M_k$. In particular, $F_N \subset M_k$. Thus we have:

Claim 2.4. *A finitely generated algebra A is NC-complete if, and only if, it is Lie complete. Similarly, A is NC-nilpotent if and only if it is Lie nilpotent.*

Following [Kap98], we will use the term NC-nilpotent for what is called Lie nilpotent in [EKM09].

Proposition 2.5. *Suppose A is NC-nilpotent. Let $\overline{S} \subset (A/M_2)$ be a finitely generated multiplicative set without zero divisors, and define $S = \pi^{-1}(\overline{S})$, where $\pi : A \rightarrow A/M_2$ is the quotient map. Then:*

- (1) S satisfies the Ore condition.
- (2) $A[S^{-1}]$ is finitely generated.
- (3) $A[S^{-1}]$ is NC-nilpotent.

Proof. Claim (1) is precisely Proposition 2.1.5 in [Kap98]. By Claim 2.4, F_\bullet and M_\bullet induce the same completion, and hence A is nilpotent for one filtration if and only if it is nilpotent for the other. Claim (3) thus follows from Theorem 2.1.6 in [Kap98].

To establish finite generation, we show that $(S + M_2)^{-1}$ is generated by A and S^{-1} . For $s \in S$ and $m \in M_2$, we express $(s + m)^{-1}$ as a geometric series,

$$\begin{aligned} (s + m)^{-1} &= s^{-1}(1 + s^{-1}m)^{-1} \\ &= s^{-1} \left(\sum_{i \geq 0} (-s^{-1}m)^i \right), \end{aligned}$$

which is seen to be a finite sum using the NC-nilpotence of A , and identity (2.1.7.2) in [Kap98]:

$$ms^{-1} = s^{-1}m + s^{-2}[s, m] + s^{-3}[s, [s, m]] + \cdots$$

□

2.2 The algebra A_\star

The algebra A/M_3 carries a commutative multiplication, in addition to the non-commutative multiplication induced from A . We have:

Proposition 2.6. *The star-product, $a \star b := \frac{1}{2}(ab + ba)$, is associative on $A_\star := (A/M_3, \star)$, and the action $a \star n := \frac{1}{2}(an + na)$, for $a \in A_\star, n \in N_k(A)$ makes each $N_k(A)$ into an A_\star -module.*

The proof of Proposition 2.6 depends on the following two lemmas:

Lemma 2.7 (Corollary 6.4 in [BJ13a]). *If i is odd, then $M_i M_j \subset M_{i+j-1}$.*

Lemma 2.8. *We have the containment $[A, [A, M_k]] \subset M_{k+1}$.*

Proof. We let $a, b, c \in A, d \in L_k$, and apply Leibniz identity:

$$[a, [b, cd]] = [a, [b, c]d] + [a, c[b, d]] = [a, [b, c]]d + [b, c][a, d] + [a, c[b, d]].$$

Applying Lemma 2.7, we have that all three terms on the right are in M_{k+1} . \square

Proof of Proposition 2.6. To see that \star is associative, we compute:

$$(a \star b) \star c - a \star (b \star c) = \frac{1}{4} ((ab + ba)c + c(ab + ba) - a(bc + cb) - (bc + cb)a) = \frac{1}{4}[b, [a, c]],$$

and thus associativity holds, modulo M_3 .

The containment $M_3 M_k \subset M_{k+2}$ of Lemma 2.7 shows the map of vector spaces $A/M_3 \otimes N_k \rightarrow N_k$ is well-defined. Associativity of the action follows from Lemma 2.8; for $n \in M_k$, we have:

$$(a \star b) \star n - a \star (b \star n) = \frac{1}{4}[b, [a, n]] = 0 \pmod{M_{k+1}}.$$

\square

Definition 2.9. We denote by A_\star the algebra $(A/M_3, \star)$, and we let $X_\star := \text{Spec} A_\star$.

Example 2.10. When $A = A_n$, the Feigin-Shoikhet isomorphism [FS07] identifies A/M_3 with the algebra $\Omega^{ev}(\mathbb{C}^n)$ of even degree differential forms on \mathbb{C}^n , equipped with the Fedosov quantized product,

$$\omega \nu := \omega \wedge \nu + d\omega \wedge d\nu.$$

In this case A_\star is the commutative algebra of even-degree differential forms, with ordinary product.

3 Lower central series components as coherent sheaves on X_\star

Throughout this section we assume that A is an NC-nilpotent algebra, with $M_N(A) = 0$. In the previous section, we endowed each $N_k(A)$ with the structure of a module for the commutative algebra A_\star , and hence obtained a quasi-coherent sheaf \widetilde{N}_k on X_\star . On the other hand, N_k also determines a pre-sheaf on X_\star by the assignment on affine opens $U_{\bar{S}} \subset X_\star \mapsto N_k(A[S^{-1}])$. The purpose of this section is to show that the two pre-sheaves obtained this way coincide; this is the content of:

Theorem 3.1. *Fix a multiplicative subset $\bar{S} \subset (A/M_2)$ without zero divisors, and let $S := \pi^{-1}(\bar{S}) \subset A$. The map,*

$$m : A_\star[S^{-1}] \otimes_{A_\star} N_k(A) \xrightarrow{\sim} N_k(A[S^{-1}]),$$

$$a \otimes n \mapsto an,$$

is an isomorphism.

Remark 3.2. It follows from [EKM09], Theorem 2.1, that $N_k(A)$ is finitely generated over A/M_3 in the case that A is a free algebra of finite rank, and thus whenever A is finitely generated. Thus each $N_k(A)$ is a coherent sheaf over X_\star .

Remark 3.3. We note that there is no loss of generality in assuming A is NC-nilpotent: we have $N_k(A) \cong N_k(A/M_{k'})$ for $k' > k$; this allows us to apply Theorem 3.1 to an arbitrary finitely generated algebra by first quotienting by sufficiently large $M_{k'}$.

The remainder of this section is devoted to proving Theorem 3.1. We will make repeated use of the following computation:

Lemma 3.4. *Let $f \in A$ invertible. Then we have: $[f^{-1}, g] = -f^{-2}[f, g] \pmod{M_3(A)}$.*

Proof. We compute:

$$\begin{aligned} 0 &= f^{-1}[f^{-1}f, g] \\ &= f^{-1}(f^{-1}[f, g] + [f^{-1}, g]f) \\ &= f^{-1}(f^{-1}[f, g] + f[f^{-1}, g] + [[f^{-1}, g], f]) \\ &= f^{-2}[f, g] + [f^{-1}, g] + f^{-1}[[f^{-1}, g], f]. \end{aligned}$$

□

Remark 3.5. In the Feigin-Shoikhet isomorphism $A/M_3 \cong \Omega^{ev}(\mathbb{C}^n)$, a commutator $[f, g]$ is sent to the two-form $2df \wedge dg$. We note that Lemma 3.4 is compatible with the Feigin-Shoikhet isomorphism, via the identity $d(\frac{1}{f}) = \frac{-1}{f^2}df$.

For elements $a_1, \dots, a_l \in A$, we denote by $M_k(a_1, \dots, a_l)$ the image of $M_k(A_l)$ under the homomorphism $A_l \rightarrow A$, given by $x_i \mapsto a_i$.

Lemma 3.6. *For $m \in M_k$, the expression*

$$[b, d][a, m] + [a, d][b, m] \tag{1}$$

lies in M_{k+2} .

Proof. The expression maps to zero under the Feigin-Shoikhet map, so it lies in $M_3(a, b, d, m)$. The space $M_3(a, b, d, m)$ is spanned by the elements:

$$m[a, [b, d]], a[b, [d, m]], [a, [b, dm]], [ab, [d, m]], [a, [bd, m]],$$

together with their permutations in the symbols a, b, d . We have $m[a, [b, d]] \in M_k M_3 \subset M_{k+2}$, by Lemma 2.7 while the remaining elements lie in M_{k+2} , by Lemma 2.8. □

Theorem 3.7. *For k odd, we have:*

$$[A, [A, S^{-1}M_k(A)]] \subset S^{-1}M_{k+2}(A) + M_{k+3}(A[S^{-1}]).$$

Proof. Let $T = S^{-1}M_{k+2}(A) + M_{k+3}(A[S^{-1}])$. We need to show that, for $a, b \in A, d \in S$, and $m \in M_k(A)$, we have $[a, [b, d^{-1}m]] = 0 \pmod T$. The proof consists of moving d^{-1} out of the nested commutator, modulo terms in $M_{k+3}(A[S^{-1}])$. We compute:

$$[a, [b, d^{-1}m]] = [a, [b, d^{-1}]m] + [a, d^{-1}[b, m]] \quad (2)$$

$$= -[a, d^{-2}[b, d]m] + [a, d^{-1}[d^{-1}, [b, d]]m] + [a, d^{-1}[b, m] + d^{-1}[a, [b, m]]] \quad (3)$$

$$= -[a, d^{-2}[b, d]m] + [a, d^{-1}[b, m] + d^{-1}[a, [b, m]]] \pmod{L_{k+3}(A[S^{-1}])} \quad (4)$$

In passing from equation (2) to (3), we have manipulated the first term as in the proof of Lemma 3.4, and applied Jacobi identity to the second term. Because k is odd, Theorem 1.3 of [BJ13a] implies that the second term in (3) lies in $[A, M_3M_k] \subset [A, M_{k+2}] \subset L_{k+3}(A[S^{-1}])$, so it is zero modulo T . The third summand in equation (4) lies in T , so may be ignored. Expanding the first summand of equation (4) with the Leibniz rule, and applying Lemma 3.4 to the second, we obtain:

$$[a, [b, d^{-1}m]] = -d^{-2}[a, [b, d]m] - [a, d^{-2}][b, d]m - d^{-2}[a, d][b, m] \pmod T. \quad (5)$$

Applying Lemma 3.4 to the second RHS term of (5) gives:

$$2d^{-3}[a, d][b, d]m \in S^{-1}M_3(A)M_k(A) \subset S^{-1}M_{k+2}(A) \subset T,$$

as in the proof of Lemma 3.6. Thus we have:

$$\begin{aligned} [a, [b, d^{-1}m]] &= -d^{-2}([a, [b, d]m] + [a, d][b, m]) \pmod T \\ &= -d^{-2}([b, d][a, m] + [a, d][b, m]) - d^{-2}[a, [b, d]]m \pmod T, \end{aligned}$$

by the Jacobi identity. By Lemma 3.6 the first expression lies in $S^{-1}M_{k+2}(A) \subset T$; by Lemma 2.7, the second expression does as well. \square

Corollary 3.8. *We have:*

1. $M_k(A[S^{-1}]) = A[S^{-1}] \otimes_A M_k(A) + M_{k+1}(A[S^{-1}])$.
2. $M_k(A[S^{-1}]) = A[S^{-1}] \otimes_A M_k(A)$.

Proof. Assertion (2) follows by repeatedly applying (1), and using that $A[S^{-1}]$ is NC-nilpotent. For (1), we consider an arbitrary generator of $M_k(A[S^{-1}])$,

$$v = a[l_1, [l_2, [\dots [l_{k-1}, l_k] \dots]],$$

where the l_i are some monomials in the generators of $A[S^{-1}]$. If each l_i lies in A , there is nothing to prove; thus, we suppose that $l_i = \tilde{l}_i f^{-1}$, for some $\tilde{l}_i \in A$, $f \in S$, and $i \in I$. By repeated use of Jacobi identity, we may assume that $i = k$.

By repeatedly applying Theorem 3.7, we may pull the f^{-1} out of the nested commutator two slots at a time, until we are left with expressions of the form:

$$\begin{aligned} &af^{-1}[l'_1, [l'_2, \dots [l'_{k-1}, l'_k] \dots]], \text{ if } k \text{ is odd, or} \\ &a[l'_1, f^{-1}[l'_2, \dots [l'_{k-1}, l'_k] \dots]], \text{ if } k \text{ is even,} \end{aligned}$$

where we have ignored any additional terms produced which lie in $M_{k+1}(A[S^{-1}])$. In the first case, there is one less inversion in the iterated commutator expression, so by induction v lies in $A[S^{-1}] \otimes_A M_k(A) + M_{k+1}(A[S^{-1}])$. In the second case, we compute, using the Jacobi identity:

$$\begin{aligned} & a[l'_1, f^{-1}[l'_2, \dots [l'_{k-1}, l'_k] \dots]] \\ &= af^{-1}[l'_1, [l'_2, \dots [l'_{k-1}, l'_k] \dots]] + a[l'_1, f^{-1}][l'_2, \dots [l'_{k-1}, l'_k] \dots] \\ &= af^{-1}[l'_1, [l'_2, \dots [l'_{k-1}, l'_k] \dots]] - af^{-2}[l'_1, f][l'_2, \dots [l'_{k-1}, l'_k] \dots] \pmod{M_{k+1}(A[S^{-1}])}. \end{aligned}$$

The second summand above lies in $M_2 M_{k-1} \subset M_k$, and both summands feature one less inversion in the nested commutator, and so by induction lie in $A[S^{-1}] \otimes_A M_k(A) + M_{k+1}(A[S^{-1}])$. \square

Proof of Theorem 3.1. Having proved Corollary 3.8, the proof echoes Theorem 2.1.6 of [Kap98]. Namely, we have an exact sequence:

$$0 \rightarrow M_{k+1} \rightarrow M_k \rightarrow N_k \rightarrow 0. \quad (6)$$

By Proposition 2.5, $A[S^{-1}]$ is a flat A -module; applying $A[S^{-1}] \otimes_A -$ we obtain:

$$0 \rightarrow A[S^{-1}] \otimes_A M_{k+1} \rightarrow A[S^{-1}] \otimes_A M_k \rightarrow A[S^{-1}] \otimes_A N_k \rightarrow 0. \quad (7)$$

Combining this with Corollary 3.8, we obtain

$$A[S^{-1}] \otimes_A N_k(A) \cong \frac{A[S^{-1}] \otimes_A M_k}{A[S^{-1}] \otimes_A M_{k+1}} \cong \frac{M_k(A[S^{-1}])}{M_{k+1}(A[S^{-1}])} \cong N_k(A[S^{-1}]). \quad (8)$$

\square

Corollary 3.9. *The completion $N_k(A)_{(\mathfrak{m}_x)}$ at any smooth point $x \in X^{sm}$ is a quotient of $N_k(\widehat{A}_n)$, where $n = \dim X$.*

Corollary 3.10. *Suppose that $X = \text{Spec}(A/M_2)$ is a zero or one dimensional scheme with finitely many non-reduced points. Then each $N_k(A)$ is a finite dimensional vector space, for $k \geq 2$.*

Proof. By Theorem 3.1, each $N_k(A)$ forms a coherent sheaf on X_\star . Over any smooth, reduced point of X , the fiber is isomorphic to $N_k(A_n) = 0$, for $n = 0, 1$; hence the sheaf is supported only over the singular and non-reduced points of X , where it is of finite rank. \square

Remark 3.11. The assumption that there are finitely many non-reduced points is necessary. For example, let $A = k\langle x, y \rangle / (y^2)$. Then the set $\{[x^l, y], l \geq 1\}$ is linearly independent in N_2 . See [CF13] for an elaboration of this example, and related examples.

Corollary 3.12. *Suppose A is the quotient of a free algebra on generators x_1, \dots, x_n in degrees d_1, \dots, d_n , by a collection of homogeneous non-commutative polynomials, so that A is graded by degree. Then the Hilbert series for each $N_k(A)$ is a rational function, with poles at d_i th roots of unity.*

Proof. This follows immediately from Hilbert's syzygy theorem for graded modules over polynomial rings. \square

4 Formal geometry and locally free algebras

Throughout this section, we assume A is a finitely generated, locally free algebra of rank n . We begin with a review of the basic constructions in formal geometry, and then give an alternative construction of $N_k(A)$ in terms of the formal geometry on X . For background, we refer to [BK04], Section 3, whose conventions we follow.

4.1 Harish-Chandra torsors

Let G be a pro-algebraic group, $G = \varprojlim G^k$. Let $\mathfrak{g} = \text{Lie}(G)$ denote the Lie algebra of G , and let $\rho : \mathcal{M} \rightarrow X$ be a G torsor, that is, a scheme \mathcal{M} over X , equipped with a G action commuting with ρ , and inducing an isomorphism $G \times \mathcal{M} \cong \mathcal{M} \times_X \mathcal{M}$.

Definition 4.1. We say a G -module N is a *pro-finite* if we have $N = \varprojlim N^k$ of finite dimensional representations N^k of the algebraic groups G^k .

Throughout this section, we take all G -modules to be pro-finite.

Definition 4.2. Given a G -module N , the *associated vector bundle* $N_{\mathcal{M}}$ on X is:

$$N_{\mathcal{M}} := \mathcal{M} \times N / \{(m \cdot g, v) \sim (m, g \cdot v)\}. \quad (9)$$

Remark 4.3. By construction, the fiber of $N_{\mathcal{M}}$ over $x \in X$ is non-canonically isomorphic to N , and we have an isomorphism,

$$\Gamma(X, N_{\mathcal{M}}) \cong \Gamma(\mathcal{M}, \mathcal{M} \times N)^G. \quad (10)$$

We have an associated G -equivariant exact sequence of sheaves on \mathcal{M} :

$$0 \rightarrow \mathcal{T}_{\mathcal{M}/X} \rightarrow \mathcal{T}_{\mathcal{M}} \rightarrow \rho^* \mathcal{T}_X \rightarrow 0 \quad (11)$$

which, by descent gives us an exact sequence of sheaves on X (known as the *Atiyah extension*)

$$0 \rightarrow \mathfrak{g}_{\mathcal{M}} \xrightarrow{i} \mathcal{E}_{\mathcal{M}} \xrightarrow{j} \mathcal{T}_X \rightarrow 0. \quad (12)$$

Here $\mathfrak{g}_{\mathcal{M}}$ denotes the bundle associated to the G -module \mathfrak{g} , as explained above. Recall that a connection on \mathcal{M} is a splitting $\theta_{\mathcal{M}} : \mathcal{E}_{\mathcal{M}} \rightarrow \mathfrak{g}_{\mathcal{M}}$. We will need to consider a more general notion of connection, taking values in a larger Lie algebra than \mathfrak{g} .

Definition 4.4. A *Harish-Chandra pair* $\langle G, \mathfrak{h} \rangle$ is a pro-algebraic group G , together with a Lie algebra \mathfrak{h} with G -action, and a compatible inclusion $\iota : \mathfrak{g} \hookrightarrow \mathfrak{h}$.

Definition 4.5. A connection $\theta_{\mathcal{M}}$ for $\langle G, \mathfrak{h} \rangle$ is a map $\theta_{\mathcal{M}} : \mathcal{E}_{\mathcal{M}} \rightarrow \mathfrak{h}_{\mathcal{M}}$ such that $\iota = \theta_{\mathcal{M}} \circ i$. The pair $(\mathcal{M}, \theta_{\mathcal{M}})$ is called a $\langle G, \mathfrak{h} \rangle$ -torsor with connection.

Definition 4.6. For G connected, we say that an \mathfrak{h} -module N is a $\langle G, \mathfrak{h} \rangle$ -module if the induced action of \mathfrak{g} is integrated to an algebraic representation of G .

Given a $\langle G, \mathfrak{h} \rangle$ -torsor with connection $(\mathcal{M}, \theta_{\mathcal{M}})$ a $\langle G, \mathfrak{h} \rangle$ -module N , the vector bundle $N_{\mathcal{M}}$ carries a connection ∇ , called the Harish-Chandra connection, given by the formula, for $s \in \Gamma(U, N_{\mathcal{M}})$ and $\xi \in \Gamma(U, \mathcal{T}_X)$:

$$\nabla_{\xi}(s) := \tilde{\xi} \cdot s - \theta_{\mathcal{M}}(\tilde{\xi}) \cdot s,$$

where $\tilde{\xi}$ is any lift of ξ to $\mathcal{E}_{\mathcal{M}}$; in fact ∇_{ξ} is independent of the choice of lift.

Remark 4.7. Given a pair $\langle G, \mathfrak{h} \rangle$, one can define a formal group H which contains G and has Lie algebra \mathfrak{h} such that H/G is the formal neighborhood of zero in $\mathfrak{h}/\mathfrak{g}$. Then there is an H -torsor associated to any G -torsor, and splitting an analogue of (12) for this H -torsor is equivalent to a connection on a $\langle G, \mathfrak{h} \rangle$ -torsor as defined above.

4.2 Formal geometry

The fundamental example of a Harish-Chandra torsor we will use is the *bundle of formal coordinate systems*, whose construction we now recall.

Let D_n denote the formal disc, $D_n = \text{spf}(\mathbb{C}[[x_1, \dots, x_n]])$; denote the unique maximal ideal by \mathfrak{m}_0 . We have the pro-algebraic group $\text{Aut}(D_n) = \varprojlim \text{Aut}^k(D_n)$, the inverse limit over k of the automorphism groups $\text{Aut}^k(D_n)$ of k -th order infinitesimal neighborhoods of the origin in \mathbb{A}^n .

Let W_n denote the Lie algebra of vector fields on D_n , and let W_n^0 denote the subalgebra of vector fields vanishing at the closed point and more generally let W_n^k denote vector fields vanishing to order $k+1$ at the closed point. In other words, W_n is the Lie algebra of derivations of $\mathbb{C}[[x_1, \dots, x_n]]$, W_n^0 is the subalgebra preserving the augmentation ideal, and W_n^k is the subalgebra $\mathfrak{m}_0^{k+1}W_n$, under the natural $\mathbb{C}[[x_1, \dots, x_n]]$ -module structure on W_n . We have an identification $W_n^0 = \text{Lie}(\text{Aut}(D_n))$.

The following proposition is well known.

Proposition 4.8. *A finite dimensional W_n^0 -module N integrates to an $\text{Aut}(D_n)$ -module if, and only if, the Euler operator $E = \sum x_i \partial_i$ acts diagonalizably with integer eigenvalues. Moreover, for such N there exists k such that the $\text{Aut}(D_n)$ -action on N factors through $\text{Aut}^k(D_n)$.*

Using the above proposition it is easy to confirm that all W_n -modules encountered in this section give rise to $\langle \text{Aut}(D_n), W_n \rangle$ -modules.

Example 4.9. Examples of pro-finite representations of $\text{Aut}(D_n)$ coming from geometry include power series rings $M = k[[x_1, \dots, x_n]]$, and more generally, the completion of a natural vector bundle at a smooth point on a variety of dimension n (see Section 4.3).

Definition 4.10. The *bundle of formal coordinate systems* \mathcal{M}_X on a smooth scheme X of dimension n consists of pairs $(x \in X, t_x : X_{(x)} \xrightarrow{\sim} D_n)$. Projection to the first factor makes \mathcal{M}_X a $\text{Aut}(D_n)$ -torsor over X .

Remark 4.11. This definition is incomplete, in that it doesn't define \mathcal{M}_X as a scheme: in fact, \mathcal{M}_X has the structure of a scheme of infinite type over X , and is locally trivial as a

$Aut(D_n)$ -torsor in the Zariski topology on X . We refer the reader to [FBZ01], Sections 9.4.4 and 11.3.3, and [BK04], Section 3.1, for complete definitions regarding the scheme structure on \mathcal{M}_X .

The action of W_n on $\mathbb{C}[[x_1, \dots, x_n]]$ defines a map,

$$\tilde{a} : W_n \times \mathcal{M}_X \rightarrow T_{\mathcal{M}_X/X},$$

descending to an isomorphism,

$$a : (W_n)_{\mathcal{M}} \xrightarrow{\sim} \mathcal{E}_{\mathcal{M}}.$$

Definition 4.12. The *bundle of flat formal coordinate systems* on X is the $\langle Aut(D_n), W_n \rangle$ -torsor, $(\mathcal{M}_X, \theta_{\mathcal{M}_X} = a^{-1})$.

Remark 4.13. Given a $\langle Aut(D_n), W_n \rangle$ -module N , and $U \subset X$ over which $N_{\mathcal{M}_X}$ trivializes, the Harish-Chandra connection $N_{\mathcal{M}_X}|_U = N_{\mathcal{M}_U}$ is given by:

$$\nabla_{\xi}(s) = ds(\xi) - \xi \circ s.$$

Thus a section $s \in \Gamma(X, N_{\mathcal{M}_X})$ is flat if the derivative, $\xi(s)$, of s along the base is simply the action of ξ in the fiber.

Definition 4.14. The globalization $\mathcal{GL}_X(N)$ to X of a $\langle Aut(D_n), W_n \rangle$ -module N is the sheaf of flat sections of the associated bundle $(N_{\mathcal{M}_X}, \nabla)$.

4.3 Natural vector bundles

A natural vector bundle is essentially a functorial assignment of a vector bundle $\mathcal{V}(X) \rightarrow X$, to every smooth X . More precisely, we have:

Definition 4.15. The category Sm_n has as objects smooth, finite-type formal schemes, of dimension n , and as morphisms étale maps.

Remark 4.16. Thus, the category Sm_n is just general enough to encompass both ordinary smooth algebraic varieties, as well as formal neighborhoods of smooth points.

Definition 4.17. The category VB_n , of vector bundles with n -dimensional base, has as objects pairs (X, V) of a smooth, finite-type formal scheme X of dimension n , and a vector bundle $V \rightarrow X$ over X . A morphism from (X, V) to (Y, W) is an étale map $f : X \rightarrow Y$, together with an isomorphism $\phi : V \rightarrow f^*(W)$.

We have a forgetful functor $F : VB_n \rightarrow Sm_n$, forgetting the vector bundle.

Definition 4.18. A natural vector bundle \mathcal{V} on n -dimensional schemes is a functorial splitting of F , i.e. a functor $G : Sm_n \rightarrow VB_n$, such that $G \circ F = \text{id}_{Sm_n}$. A morphism of natural vector bundles is a natural transformation of the corresponding functors. This defines the category, \mathcal{NVB}_n , of natural vector bundles.

Semi-simple examples of natural vector bundles include: trivial vector bundles, tangent and cotangent bundles, differential forms, polyvector fields, and more generally any tensor bundle built from tangent and cotangent bundles by Schur functors. Non-semi-simple examples include bundles of polynomial differential operators of order $\leq k$, for any k .

Definition 4.19. Let $i_0 : \{0\} \hookrightarrow D_n$ denote the inclusion of the origin into D_n . The *standard fiber* $St(\mathcal{V})$ of $\mathcal{V} \in \mathcal{NVB}_n$ is $St(\mathcal{V}) = i_0^*(\mathcal{V}(D_n))$.

Applying Definition 4.18, each automorphism $g : D_n \rightarrow D_n$ induces an isomorphism $\phi_g : \mathcal{V}(D_n) \rightarrow g^*\mathcal{V}(D_n)$. Since g preserves the closed point, the assignment $g \mapsto \phi_g$ defines an action of $Aut(D_n)$ on $St(\mathcal{V})$. Thus we have the *standard fiber* functor:

$$St : \mathcal{NVB}_n \rightarrow Aut(D_n)\text{-mod}_f,$$

to the category of finite dimensional algebraic $Aut(D_n)$ -modules.

Conversely, given a finite dimensional algebraic $Aut(D_n)$ -module N , the assignment to each smooth scheme X of its associated bundle $N_{\mathcal{M}_X}$ determines a natural vector bundle, $Assoc(N)$.

Proposition 4.20. *The functors St and $Assoc$ are mutually inverse equivalences of categories.*

Proof. Let $M \in Aut(D_n)\text{-mod}_f$. We have a natural isomorphism $M \xrightarrow{\sim} St \circ Assoc(M)$, which simply identifies M with the fiber of the associated bundle over the closed point.

For $\mathcal{V} \in \mathcal{NVB}_n$, we now construct a natural isomorphism $\mathcal{V} \rightarrow Assoc(St(\mathcal{V}))$. Set $\mathcal{G} = Assoc \circ St(\mathcal{V})$. Given $X \in Sm_n$, the data of a map $\Psi : \mathcal{G}(X) \rightarrow \mathcal{V}(X)$ is equivalent to that of a G -equivariant map $\tilde{\Psi}$ as shown:

$$\begin{array}{ccccc} \mathcal{M}_X & & \pi^*\mathcal{G}(X) & \xrightarrow{\tilde{\Psi}} & \pi^*\mathcal{V}(X) \\ \downarrow \pi & & \downarrow & & \downarrow \\ X & & \mathcal{G}(X) & \xrightarrow{\Psi} & \mathcal{V}(X) \end{array}$$

with dashed arrows indicating the descent functor (i.e. taking $Aut(D_n)$ -invariants). Note that any point $(x, \varphi) \in \mathcal{M}_X$ gives a map $\varphi_x : St(\mathcal{V}) \xrightarrow{\sim} \mathcal{V}_x$, the fiber of \mathcal{V} at $x \in X$. Now let s be a section of $\pi^*\mathcal{G}(X)$ over \mathcal{M}_X . We define:

$$\left(\tilde{\Psi}(s)\right)(x, \varphi) := (x, \varphi_x(s(x))).$$

It is easy to see that $\tilde{\Psi}$ is a G -equivariant isomorphism, hence it induces an isomorphism Ψ . \square

Globalizations of certain W_n -equivariant \mathcal{O}_{D_n} -modules provide an important source of natural vector bundles. We outline the construction now.

Definition 4.21. Let $Vec(D_n/W_n)$ denote the category of finite rank $\mathbb{C}[[x_1, \dots, x_n]]$ -modules with a compatible $\langle Aut(D_n), W_n \rangle$ -module structure.

Remark 4.22. Equivalently, an object of $Vec(D_n/W_n)$ is a finite rank $\mathbb{C}[[x_1, \dots, x_n]]$ -module M with compatible W_n -action, such that the Euler operator E acts diagonalizably with integer eigenvalues on the associated graded module, $gr(M) = \bigoplus_k \mathfrak{m}_0^k M / \mathfrak{m}_0^{k+1}$.

We note that the globalization functor \mathcal{GL}_X applied to the W_n -module $\mathbb{C}[[x_1, \dots, x_n]]$ yields jets of the structure sheaf \mathcal{O}_X , compatibly with the action of \mathcal{O} on any $M \in Vec(D_n/W_n)$. This endows the globalization $\mathcal{GL}_X(M)$ of $M \in \mathcal{W}_{\mathcal{O}}^{int}$ -mod with the structure of a vector bundle. The naturality of the torsor \mathcal{M}_X under étale morphisms gives a functor:

$$\mathcal{GL} : Vec(D_n/W_n) \rightarrow \mathcal{NVB}_n.$$

Conversely, given a natural vector bundle \mathcal{V} , the evaluation $ev_{D_n} : \mathcal{V} \mapsto \mathcal{V}(D_n)$ is naturally an object of $Vec(D_n/W_n)$.

Proposition 4.23. *The functors ev_{D_n} and \mathcal{GL} define mutually inverse equivalences of categories.*

Proof. A natural isomorphism $\mathcal{GL} \circ ev_{D_n}(\mathcal{V}) \rightarrow \mathcal{V}$ can be constructed as in the proof of 4.20. For $M \in Vec(D_n/W_n)$ -mod, we construct a natural isomorphism $M \rightarrow ev_{D_n} \circ \mathcal{GL}(M)$, as follows. Let $m \in M$. We have a canonical trivialization of the associated bundle $M_{\mathcal{M}_{D_n}}$ over D_n . For $m \in M$, we define a section $f_m \in \Gamma(D_n, \mathcal{GL}_{D_n}(M))$, by:

$$f_m(y_1, \dots, y_n) = e^{\sum_i y_i \partial_i} \cdot m = \sum_I \frac{1}{I!} y^I (\partial_I m).$$

On the right hand side, ∂_i acts on $m \in M$ via the W_n -action, I ranges over all multi-indices $I = (i_1, \dots, i_n)$ with each $i_k \geq 0$, $y^I = y^{i_1} \dots y^{i_n}$, and $\partial_I = \partial_{i_1} \dots \partial_{i_n}$. In other words, f_m is the unique flat section taking value m at the closed point. The correspondence $m \mapsto f_m$ is clearly injective and surjective, since a flat section on D_n is uniquely determined by its value at the origin. □

In summary, we have the following categories, and functors between them:

$$Vec(D_n/W_n) \begin{array}{c} \xrightarrow[\sim]{\mathcal{GL}} \\ \xleftarrow{ev_{D_n}} \end{array} \mathcal{NVB}_n \begin{array}{c} \xrightarrow[\sim]{St} \\ \xleftarrow{Assoc} \end{array} Aut(D_n)\text{-mod}_f \quad (13)$$

Remark 4.24. By Proposition 4.8, the equivalences outlined above reduce the problem of studying natural vector bundles to that of studying finite dimensional representations of the algebraic groups $Aut^k(D_n)$. These groups are unipotent extensions of the group GL_n ; as such their finite dimensional irreducible representations are all pulled back from GL_n .

4.4 Lower central series of locally free algebras

Having reviewed the basic setup for formal geometry and natural vector bundles, we now explain the relation to lower central series.

Lemma 4.25. *Let \widehat{A} be a local ring, non-canonically isomorphic to \widehat{A}_n . Then any isomorphism $\psi : \widehat{A}/M_2 \xrightarrow{\sim} \mathbb{C}[[x_1, \dots, x_n]]$ induces a canonical isomorphism $\widetilde{\psi} : N_k(\widehat{A}) \xrightarrow{\sim} N_k(\widehat{A}_n)$. Moreover, for $g \in \text{Aut}(D_n)$, we have $\widetilde{g \circ \psi} = \widetilde{g} \circ \widetilde{\psi}$.*

Proof. We have the Feigin-Shoikhet natural isomorphisms $\xi : (\widehat{A}/M_3)_\star \cong \Omega^{ev}(\widehat{A}/M_2)$. Recall that $\Omega^{ev}(-)$ is functorial w.r.t isomorphisms: an isomorphism $f : A_1 \xrightarrow{\sim} A_2$ of commutative algebras induces an isomorphism $\Omega^{ev}(f) : \Omega^{ev}(A_1) \xrightarrow{\sim} \Omega^{ev}(A_2)$. Thus, given ψ , we have the natural isomorphism $\overline{\Psi}$, defined as the composition:

$$\begin{array}{ccc} (\widehat{A}/M_3)_\star & \xrightarrow[\xi]{\sim} & \Omega^{ev}(\widehat{A}/M_2) \\ \downarrow \overline{\Psi} & & \downarrow \Omega^{ev}(\psi) \\ (\widehat{A}_n/M_3)_\star & \xleftarrow[\xi^{-1}]{\sim} & \Omega^{ev}(\widehat{A}_n/M_2) \end{array}$$

Let $\Psi : \widehat{A} \rightarrow \widehat{A}_n$ denote an arbitrary isomorphism lifting of $\overline{\Psi}$. We claim that $\Psi|_{N_k(A)}$ is independent of the choice of lift. Suppose that Ψ' is another lift; then $\Psi'(a) - \Psi(a) \in M_3(\widehat{A}_n)$, for all $a \in \widehat{A}$. The containments $M_3 M_k \subset M_{k+2}$, and $[A, M_3] \subset L_4$ (Theorem 1.3 from [BJ13a]) imply that Ψ and Ψ' agree modulo $M_{k+1}(\widehat{A}_n)$:

$$\begin{aligned} \Psi(a_0[a_1, [a_2, \dots, [a_{k-1}, a_k]] \dots) &= \Psi(a_0)[\Psi(a_1), \dots [\Psi(a_{k-1}), \Psi(a_k)] \dots] \\ &= \Psi'(a_0)[\Psi'(a_1), \dots [\Psi'(a_{k-1}), \Psi'(a_k)] \dots] \pmod{M_{k+1}}. \end{aligned}$$

Thus, we may set $\widetilde{\psi} := \Psi$. The second part of the claim follows from the naturality in constructing $\overline{\Psi}$. \square

Similarly, any $\chi \in W_n$ induces an endomorphism of $N_k(\widehat{A}_n)$, compatible with the identification $W_n^0 \cong \text{Lie}(\text{Aut}(D_n))$. As a consequence, each component $N_k(\widehat{A}_n)$ is a module for the pair $(\text{Aut}(D_n), W_n)$. Thus, we have the sheaf $\mathcal{GL}_X(N_k(\widehat{A}_n))$ on X . The main result of this section is the following:

Theorem 4.26. *We have $\widetilde{N}_k(A) \cong \mathcal{GL}_X(N_k(\widehat{A}_n))$ as vector bundles on X . In particular, we have $N_k(A) \cong \Gamma(X, \mathcal{GL}_X(N_k(\widehat{A}_n)))$, as $U(\text{Vect}(X)) \ltimes \mathcal{O}_X$ -modules.*

Applying the construction of Proposition 2.6 we may regard $N_k(\widehat{A}_n)$ as an object of $\mathcal{W}_{n, \mathcal{O}}^{int}\text{-mod}$. Thus, combining Theorem 4.26 and the existence of Kapranov's locally free lifts $A(X)$ of any smooth X , we have:

Corollary 4.27. *The assignment $X \mapsto N_k(A(X))$, determines a natural vector bundle.*

Remark 4.28. It should be noted here (see [Kap98], Remark 1.6.4) that the thickenings $A(X)$ constructed by Kapranov are not themselves functorial in X , only the components $N_k(A(X))$ are, by Theorem 4.26.

Proof of Theorem 4.26. Let $p : \mathcal{M}_X \rightarrow X$ denote the canonical projection, and let $U \subset X$ open. For each $x \in X$, we have canonical maps,

$$\rho_x : \widetilde{N}_k(U) \rightarrow (\widetilde{N}_k)_{(\mathfrak{m}_x)} \xrightarrow{\sim} N_k(A_{(\mathfrak{m}_x + M_2)}), \quad (14)$$

composing the restriction to stalks with the isomorphism of Corollary 3.9. We define:

$$\begin{aligned}\varphi : \widetilde{N}_k(U) &\rightarrow \Gamma(p^{-1}(U), \mathcal{M}_X \times N_k(\widehat{A}_n)), \\ \varphi(f)(x, \psi) &= \widetilde{\psi}(\rho_x(f)),\end{aligned}$$

where $\widetilde{\psi}$ is the lift of ψ constructed in Lemma 4.25.

Proposition 4.29. *The section $\varphi(f)$ is \widehat{G} -equivariant, for all $f \in \widetilde{N}_k(U)$.*

Proof. We compute:

$$\begin{aligned}g \cdot (\varphi(f)(x, \psi)) &= g \cdot (\widetilde{\psi}(\rho_x(f))) \\ &= \widetilde{g}(\widetilde{\psi}(\rho_x(f))) \\ &= \widetilde{(g \circ \psi)}(\rho_x(f)) \\ &= \varphi(f)(x, g \circ \psi).\end{aligned}$$

□

By descent, $\varphi(f)$ defines a section in $\Gamma(U, N_X) \cong \Gamma(p^{-1}(U), \mathcal{M}_X \times N_k(\widehat{A}_n))^{\widehat{G}}$, which we also call $\varphi(f)$.

Proposition 4.30. *The section $\varphi(f)$ is flat, for all $f \in \widetilde{N}_k(U)$.*

Proof. A vector field ξ gives rise to a family ξ_x of derivations of $\mathcal{O}(X_{(\mathfrak{m}_x)})$ at each smooth point $x \in X$. By definition of the Harish-Chandra connection, $\phi(f)$ is flat, and only if, the action of ξ_x on each $N_k(A_{(x)})$ agrees with the action of ξ on $N_k(A)_{(x)}$. Thus flatness of $\phi(f)$ follows from the compatibility of the isomorphism of Theorem 3.1 with derivations. □

Thus, the map φ defines a map of sheaves, $\varphi : \widetilde{N}_k(A) \rightarrow \mathcal{GL}_X(N_k(\widehat{A}_n))$. It suffices to check that φ is an isomorphism on stalks, which follows from Corollary 3.9. □

It now follows from Remark 4.24 that the data of the lower central series components N_k , on any locally free algebra A of rank n , are completely determined by the standard fiber of $N_k(A_n)$. This is computationally useful even for $N_k(A_n)$ itself, as the examples below illustrate.

Example 4.31. A basis for the standard fiber $St(N_3(A_n)) = N_3/A_n^+N_3$ is:

$$\{[x_i, [x_i, x_j]], [x_i, x_j][x_k, x_l]\}.$$

We conclude that $St(N_3(A_2))$ is the W_n^0 -module

$$0 \rightarrow V_{(2,2)} \rightarrow St(N_3(A_2)) \rightarrow V_{(2,1)} \rightarrow 0.$$

The sequence is not split, as $x_2^2\partial_2[x_1, [x_1, x_2]] = 2[x_1, x_2]^2$, modulo $A_n^+N_3$. This may be contrasted with [EKM09], Proposition 5.3, which derives a *split* short exact sequence:

$$0 \rightarrow \mathcal{F}_{(2,2)} \rightarrow N_3(A_2) \rightarrow \mathcal{F}_{(2,1)} \rightarrow 0$$

of W_n -modules, where $\mathcal{F}_{(m,n)}$ denotes the tensor field module co-induced from the \mathfrak{gl}_2 -module $V_{(m,n)}$. However, the splitting does not commute with the action of $\mathcal{O}_{\mathbb{A}^2}$.

Example 4.32. In [Ker13], Theorem 1.1, a bound is established on the degree $|\lambda|$ of tensor field modules \mathcal{F}_λ appearing in a W_n composition series of $N_k(A_n)$:

$$|\lambda| \leq \begin{cases} 2k - 2 & k \text{ odd}; \\ 2k - 2 + 2\lfloor \frac{n-2}{2} \rfloor & k \text{ even}. \end{cases}$$

Kerchev's bound can be re-interpreted as a bound on the degree $|\lambda|$ of the \mathfrak{gl}_n -representation V_λ appearing in $St(N_k(A_n))$, which can be established simply by bounding the degree of elements of a spanning set. Using [BJ13a], Corollary 1.5 it is easy to show that $St(N_k(A))$ is spanned by elements of the form,

$$a \star [l_1, [\cdots [l_{k-1}, l_k] \cdots]],$$

where l_1, l_k have degree at most one, l_2, \dots, l_{k-1} have degree at most 2, and a is a product of simple brackets $[x_i, x_j]$. If k is odd, then $M_2 L_k \subset M_{k+1}$ in that case, proving the bound. If k is even, a may be non-trivial, but we may assume $a[l_{k-1}]l_k \neq 0 \pmod{M_3}$; otherwise:

$$\begin{aligned} a \star [l_1, [\cdots, [l_{k-1}, l_k] \cdots]] &= [a \star l_1, [\cdots, [l_{k-1}, l_k] \cdots]] \\ &= [l_1, [a \star l_2, [\cdots, [l_{k-1}, l_k] \cdots]] \\ &= \cdots \\ &= [l_1, [\cdots, a \star [l_{k-1}, l_k]] \cdots]. \end{aligned}$$

by repeated application of Kerchev's Lemma 2.1.

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